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Sublattices of the Polynomial Time Degrees*

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We show that any countable distributive lattice can be embedded in any interval of polynomial time degrees. Furthermore the embeddings can be chosen to preserve the least or the greatest element. This holds for both polynomial time bounded many–one and Turing reducibilities, as well as for all of the common intermediate reducibilities. © 1985 Academic Press, Inc.

INTRODUCTION

Cook (1971) and Karp (1972) introduced the polynomial time bounded counterparts to the two most important recursive reducibility notions, namely polynomial time Turing (p–T) and many–one (p–m) reducibilities, respectively. While Cook's p–T-reducibility seems to be the most natural and general efficient reducibility notion, Karp's stronger p–m-reducibility proved to be of particular value for classifying problems in $\mathcal{NP}$. Ladner (1973, 1975) was the first to study the structure of the polynomial time (p–) degrees induced by these reducibility notions on the recursive sets. He showed that, for both notions, the p-degrees form an upper semilattice but not a lattice, that the partial ordering of p-degrees is dense, that every nonzero p-degree splits, i.e., is the join of two lesser ones, and that minimal pairs of p-degrees exist, i.e., that there are incomparable p-degrees **a** and **b** with infimum **0**, **0** the p-degree of the class $\mathcal{P}$ of (deterministically) polynomial time computable sets. Moreover, Ladner proved that, under the hypothesis of $\mathcal{P} \neq \mathcal{NP}$, $\mathcal{NP}$ will consist of infinitely many p-degrees.

Interesting extensions of Ladner's results were obtained by Mehlhorn (1974, 1976) who proved among others that any countable partially ordered set can be embedded in any interval of p-degrees. Landweber, Lipton, and Robertson (1981) and Chew and Machtey (1981) refined Ladner's diagonalization technique, thus simplifying and extending some of Ladner's results. For instance they showed that every nonzero p-degree bounds a minimal pair.

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Here we continue the investigation into the structure of the polynomial degrees. We prove a quite general lattice embedding theorem for the p-degrees which subsumes and extends most of the previously obtained results on the structure of the polynomial time degrees. In particular we show that any countable distributive lattice can be embedded in any interval of polynomial time degrees by maps which preserve the greatest or least element, respectively. Moreover the embeddings can be chosen to be incomparable with any finite (in fact, any recursively presentable) class of intermediate degrees. Since most of the results on polynomial time degrees in the literature can be viewed as embedding results for certain simple finite distributive lattices, we obtain these results as immediate corollaries. As we will point out our embedding results do not only hold for Cook's and Karp's reducibilities but also for the intermediate reducibilities introduced by Ladner, Lynch, and Selman (1975). Moreover we obtain similar results for the degree classes of sets with certain interesting structural properties where these classes share a certain uniformity property. We look at the following examples: non-selfdual sets, i.e., sets which cannot be p–m-reduced to their complements, non–p-mitotic sets (cf. Ambos-Spies, 1984a) and Δ_2^p sets (cf. Schöning, 1983).

After some preliminaries in Section 1, in Section 2 we summarize some facts on recursively presentable classes of recursive sets which we need for our proofs. In Section 3 we use the diagonalization technique of Landweber, Lipton, and Robertson (1981) and Chew and Machtey (1981) to prove two lemmas which provide the tools for preserving joins and meets in our lattice embeddings. Section 4 contains the main theorems. Finally, in Section 5 we apply our embedding results to the *p*-degrees of sets with special properties.

1. Preliminaries

Let Σ be a finite alphabet which contains the letters 0 and 1, and let Σ^* denote the set of (finite) strings over Σ. We denote elements of Σ^* by lower case letters from the end of the alphabet, while capital letters denote recursive subsets of Σ^*. $|x|$ is the length of x and $|A|$ is the cardinality of A. $<$ is the natural ordering on Σ^*. In our notation we do not distinguish between a set and its characteristic function. So $x \in A$ iff $A(x)=1$ and $x \notin A$ iff $A(x)=0$. $\bar{A}$ denotes the complement $\Sigma^* - A$ of A, xA is the set $\{xy\colon y \in A\}$ and $A \oplus B = 0A \cup 1B$. We write $A =^* B$ if the symmetrical difference $(A-B) \cup (B-A)$ is finite and $A \subseteq^* B$ if $A' \subseteq B$ for some $A' =^* A$. $\mathbb{N}$ is the set of natural numbers. Lower case letters from the middle of the alphabet denote elements of $\mathbb{N}$, lower case greek letters denote recursive subsets of $\mathbb{N}$.

$\mathscr{P}(\mathscr{NP})$ is the class of subsets of Σ^* which can be (non)deterministically computed in polynomial time. $\mathscr{PF}$ is the set of polynomial time computable functions from Σ^* to Σ^*, and $\mathscr{P}_{\mathbb{N}}$ is the class of subsets of $\mathbb{N}$ which are polynomial time computable (with respect to unary representation). Let $\{P_n: n \in \mathbb{N}\}$ and $\{f_n: n \in \mathbb{N}\}$ be recursive enumerations of $\mathscr{P}$ and $\mathscr{PF}$, respectively. Note that there is a canonical embedding of $\mathscr{P}_{\mathbb{N}}$ into $\mathscr{P}$ induced by the function $f(n) = 0^n$ which identifies $\mathbb{N}$ with $\{0\}^*$. So $\mathscr{P}_{\mathbb{N}}$ is isomorphic to the class of languages in $\mathscr{P}$ over the single letter alphabet $\{0\}$.

Confusing notation, we use $\langle\ ,\ \rangle$ to denote polynomial time computable and invertible bijections from $\mathbb{N} \times \Sigma^*$ to Σ^* and from $\mathbb{N}^n$ to $\mathbb{N}$ $(n \geqslant 2)$. $A^{(n)} = \{x: \langle n, x\rangle \in A\}$.

A is p–m (many–one)-reducible to B $(A \leqslant_m^p B)$ via $f \in \mathscr{PF}$, if $\forall x$ $(A(x) = B(f(x)))$, i.e., $A = f^{-1}(B)$. A is p–T (Turing)-reducible to B, if there is a polynomial-time bounded oracle Turing machine $M(X)$ such that $\forall x$ ($x \in A$ iff $M(B)$ accepts x). We write $M(B)(x) = 1$ if $M(B)$ accepts x and $M(B)(x) = 0$ otherwise. So $A \leqslant_T^p B$ via M iff $\forall x$ $(A(x) = M(B)(x))$. Besides the just introduced polynomial-time bounded versions of many–one and Turing reducibility we will also consider the polynomial versions of the following intermediate reducibilities: one-question truth-table (1–tt), bounded truth-table (b–tt), conjunctive (c), disjunctive (d), and truth-table (tt). For definitions of and basic results on these concepts we refer the reader to Ladner, Lynch, and Selman (1975). The following relations hold among these reducibility notions: $\leqslant_m^p \to \leqslant_{1-tt}^p \to \leqslant_{btt}^p \to \leqslant_{tt}^p \to \leqslant_T^p$ and $\leqslant_m^p \to \leqslant_c^p(\leqslant_d^p) \to \leqslant_{tt}^p$. In the following r stands for any of the above reducibilities.

We write $A =_r^p B$ if $A \leqslant_r^p B$ and $B \leqslant_r^p A$, $A <_r^p B$ if $A \leqslant_r^p B$ but not $A =_r^p B$, and $A \nleqslant_r^p B$ $(A \neq_r^p B)$ if not $A \leqslant_r^p B$ $(A =_r^p B)$. The p–r-degree of A is defined by $\deg_r^p(A) = \{B: B =_r^p A\}$. The partial ordering on the p–r-degrees induced by $\leqslant_r^p$ is denoted by $\leqslant$. In this paper we only consider p–r-degrees of recursive sets and we denote these degrees by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,.... Classes of p–r-degrees of recursive sets are denoted by $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$,.... We let $\mathbf{A}(\leqslant \mathbf{a}) = \{\mathbf{b} \in \mathbf{A}: \mathbf{b} \leqslant \mathbf{a}\}$, $[\mathbf{b}, \mathbf{a}] = \{\mathbf{c}: \mathbf{b} \leqslant \mathbf{c} \leqslant \mathbf{a}\}$, and $(\mathbf{b}, \mathbf{a}) = \{\mathbf{c}: \mathbf{b} < \mathbf{c} < \mathbf{a}\}$.

$\mathbf{R}_r^p$ $(\mathbf{NP}_r^p)$ is the class of p–r-degrees of recursive $(\mathscr{NP})$ sets. Note that there is a least p–r-degree $\mathbf{0}$, the degree consisting of the class $\mathscr{P}$ (in case of p–m-reducibility we systematically ignore the sets $\varnothing$ and Σ^* which constitute their own p–m-degrees), and that the supremum $\mathbf{a} \cup \mathbf{b}$ of any two p–r-degrees exists, namely $\deg_r^p(A) \cup \deg_r^p(B) = \deg_r^p(A \oplus B)$. As Ladner (1975) has shown, however, the infimum $\mathbf{a} \cap \mathbf{b}$ of two p-degrees does not always exist. So the p–r-degrees of recursive sets form an upper semilattice but not a lattice. For further basic results on the structure of the polynomial degrees we refer the reader to Ladner (1975).

2. Recursively Presentable Classes

Landweber *et al.* (1981) have observed that the notion of a recursively presentable class of recursive sets plays an important role in the study of polynomial time degrees. Here we review this and a related notion, and summarize some simple but important facts on recursively presentable classes. Some of the results presented below are taken from the literature (Landweber *et al.* (1981), Regan (1983), Schmidt (1984), and Schöning (1982)). In the following we will use the results of this section without giving explicit references.

A class $\mathscr{C}$ of recursive sets is *recursively presentable* (*r.p.*) or *uniformly recursive* if $\mathscr{C}$ is empty or there is a recursive set U such that

$$\mathscr{C} = \{U^{(e)} : e \in \mathbb{N}\}.$$

U is called a *universal set* for $\mathscr{C}$. A class $\mathscr{C}$ is *closed under finite variants* (*c.f.v.*) if, for $A \in \mathscr{C}$ and $B =^* A$, $B \in \mathscr{C}$. A class $\mathbf{C}$ of p–r-degrees is *recursively presentable* if there is an r.p. class $\mathscr{C}$ of recursive sets such that $\mathbf{C} = \{\deg_r^p(C) : C \in \mathscr{C}\}$.

In the literature recursively presentable classes are usually required to be nonempty. Inclusion of the empty class here is purely for convenience. Note that any finite class of recursive sets is r.p. and that the polynomial degrees of the members of an r.p. class are bounded, namely $\forall C \in \mathscr{C}\ (C \leqslant_m^p U)$, U some universal set for $\mathscr{C}$.

The following lemma summarizes some simple properties of r.p. classes which we will use later.

2.1. Lemma. *Let $\mathscr{C}_0, \mathscr{C}_1$ be r.p. classes of recursive sets.*

(a) *$\mathscr{C}_0 \cup \mathscr{C}_1$ i r.p. Moreover if $\mathscr{C}_0, \mathscr{C}_1$ are closed under finite variants then so is $\mathscr{C}_0 \cup \mathscr{C}_1$.*

(b) *If $\mathscr{C}_0, \mathscr{C}_1$ are c.f.v. then $\mathscr{C}_0 \cap \mathscr{C}_1$ is r.p. and c.f.v.*

(c) *The following classes are r.p. and c.f.v.:*

$$\mathscr{D}_r(\mathscr{C}_0) = \{A : \exists C \in \mathscr{C}_0 (A =_r^p C)\}$$

$$[\mathscr{C}_0, \mathscr{C}_1]_r = \{A : \exists C_0 \in \mathscr{C}_0\, \exists C_1 \in \mathscr{C}_1 (C_0 \leqslant_r^p A \leqslant_r^p C_1)\}.$$

(d) *$\mathscr{C}_0^{\mathrm{Fin}} = \{C : \exists C' \in \mathscr{C}_0\, (C =^* C')\}$ is r.p. and c.f.v. In particular the classes of finite sets $\{\varnothing\}^{\mathrm{Fin}}$ and cofinite sets $\{\Sigma^*\}^{\mathrm{Fin}}$ are r.p. and c.f.v.*

Proof. W.l.o.g. we may assume that $\mathscr{C}_0$ and $\mathscr{C}_1$ are nonempty, say U_0, U_1 are universal for $\mathscr{C}_0, \mathscr{C}_1$, respectively.

(a) The set U defined by

$$\langle 2n+i, x\rangle \in U \text{ iff } \langle n, x\rangle \in U_i \qquad (i \leqslant 1, n \in \mathbb{N}, x \in \Sigma^*)$$

is universal for $\mathscr{C}_0 \cup \mathscr{C}_1$. The second part of the claim is obvious.

(b) W.l.o.g. $\mathscr{C}_0 \cap \mathscr{C}_1 \neq \varnothing$, say $C \in \mathscr{C}_0 \cap \mathscr{C}_1$. Define a recursive set V by $V^{(\langle m,n\rangle)}(x) = U_0^{(m)}(x)$ if $\forall y < x\ (U_0^{(m)}(y) = U_1^{(n)}(y))$, and $V^{(\langle m,n\rangle)}(x) = C(x)$ otherwise $(m, n \in \mathbb{N}, x \in \Sigma^*)$. Then $V^{(\langle m,n\rangle)} = U_0^{(m)}$ if $U_0^{(m)} = U_1^{(n)}$ and $V^{(\langle m,n\rangle)} =^* C$ otherwise. So V is universal for $\mathscr{C}_0 \cap \mathscr{C}_1$.

(c) The proofs for the two classes $\mathscr{D}_r(\mathscr{C}_0)$ and $[\mathscr{C}_0, \mathscr{C}_1]_r$ and for the various reducibility notions are very similar (Obviously the classes are c.f.v. since any p–r-degree has this closure property). So we only show that $[\mathscr{C}_0, \mathscr{C}_1]_m$ is recursively presentable.

W.l.o.g. assume $[\mathscr{C}_0, \mathscr{C}_1]_m \neq \varnothing$, say $D \in [\mathscr{C}_0, \mathscr{C}_1]_m$. Let $\{f_n : n \in \mathbb{N}\}$ be a recursive enumeration of $\mathscr{PF}$. For $m = \langle i, j, k, l\rangle$ we inductively define a recursive set E_m by $E_m(x) = U_1^{(j)}(f_l(x))$ if

$$\forall y < x[(f_k(y) < x \to U_0^{(i)}(y) = E_m(f_k(y))) \quad \text{and} \quad E_m(y) = U_1^{(j)}(f_l(y))]$$

and $E_m(x) = D(x)$ otherwise. Note that either $U_0^{(i)} \leqslant_m^p E_m$ via f_k and $E_m \leqslant_m^p U_1^{(j)}$ via f_l or $E_m =^* D$. So in either case $E_m \in [\mathscr{C}_0, \mathscr{C}_1]_m$, and $[\mathscr{C}_0, \mathscr{C}_1]_m = \{E_{\langle i,j,k,l\rangle} : i, j, k, l \in \mathbb{N}\}$. Since the sets E_m are uniformly recursive, so is the set W defined by $W^{(m)} = E_m$, whence W is universal for $[\mathscr{C}_0, \mathscr{C}_1]_m$.

(d) Let x_n denote the nth string w.r.t. the natural ordering of Σ^*. Then the recursive set V, defined by $x_n \in V^{(k)}$ iff the binary representation of k has length $\geqslant n$ and the nth digit (from the right) equals one, is universal for $\{\varnothing\}^{\mathrm{Fin}}$. So W, defined by $W^{(\langle k,m,n\rangle)}(x) = U_0^{(k)}(x)$ if $|x| \geqslant m$ and $W^{(\langle k,m,n\rangle)}(x) = V^{(n)}(x)$ otherwise, enumerates all finite variants of sets in $\mathscr{C}_0$, i.e., W is universal for $\mathscr{C}_0^{\mathrm{Fin}}$.

Note that, by Lemma 2.1(c), a class $\mathbf{C}$ of p–r-degrees is recursively presentable iff $\{C : \exists \mathbf{c} \in \mathbf{C}\ (C \in \mathbf{c})\}$ is recursively presentable. Moreover, any interval of p–r-degrees is r.p. In particular, for any recursive set A, $\{B : B =_r^p A\}$ and $\{B : B \leqslant_r^p A\}$ are r.p. and c.f.v. So all complexity classes, like $\mathscr{P}$, $\mathscr{NP}$, $co\mathscr{NP}$, PSPACE, which posses complete sets with respect to $\leqslant_m^p$ and which are downwards closed under $\leqslant_m^p$ are recursively presentable and closed under finite variants.

The following notion extends recursive presentability to unbounded classes of recursive sets.

2.2. Definition. A class $\mathscr{C}$ of recursive sets is *locally recursively presentable* (*l.r.p*) if $\mathscr{C} \cap \mathscr{D}$ is r.p. for any r.p. and c.f.v. class $\mathscr{D}$.

By Lemma 2.1(b), any c.f.v. and r.p. class is l.r.p. too. There are c.f.v. and l.r.p. classes, however, which fail to be r.p., e.g., the class of all recursive sets.

2.3. LEMMA. *Let $\mathscr{C}$ be a class which is closed under finite variants.*

(a) *$\mathscr{C}$ is r.p. iff $\mathscr{C}$ is l.r.p. and bounded, i.e., for some recursive set A, $\forall C \in \mathscr{C}$ $(C \leqslant_m^p A)$.*

(b) *$\mathscr{C}$ is l.r.p. iff $\mathscr{C} \cap \{C: C \leqslant_m^p A\}$ is r.p. for any recursive set A.*

Proof. (a) As we have observed above, any c.f.v. and r.p. class is bounded and l.r.p. On the other hand if $\mathscr{C}$ is a c.f.v. and l.r.p. class which is bounded, say by the recursive set A, then $\mathscr{C} = \mathscr{C} \cap \mathscr{D}$ for the r.p. and c.f.v. class $\mathscr{D} = \{B: B \leqslant_m^p A\}$. So by local recursive presentability, $\mathscr{C}$ is also r.p.

(b) For a proof of the nontrivial direction assume that $\mathscr{C} \cap \{C \leqslant_m^p A\}$ is r.p. for any recursive set A, and let $\mathscr{D}$ be any c.f.v. and r.p. class. Then for a recursive set A bounding $\mathscr{D}$,

$$\mathscr{C} \cap \mathscr{D} = (\mathscr{C} \cap \{B: B \leqslant_m^p A\}) \cap \mathscr{D}.$$

So $\mathscr{C} \cap \mathscr{D}$ is r.p. by hypothesis and Lemma 2.1(b).

3. COMPLEXITY BOUNDED DIAGONALIZATION

In this section we use Landweber, Lipton, and Robertson's (1981) diagonalization technique to prove two diagonalization lemmas, which will be used for preserving joins and meets, respectively, in our embedding proof in the next section.

We start with some notions introduced by Landweber *et al.* (1981). Let $f: \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) > n$ for each n. The *nth iteration* f^n of f is inductively defined by $f^0(m) = m$ and $f^{n+1}(m) = f(f^n(m))$. The set

$$I_n^f = \{x \in \Sigma^*: f^n(0) \leqslant |x| < f^{n+1}(0)\}$$

is called the $(n+1)$st *f-interval.* Since $f(n) > n$, $f^n(0) > f^{n+1}(0)$ whence the f-intervals give a partition of Σ^*, i.e., $\Sigma^* = \bigcup\{I_n^f: n \in \mathbb{N}\}$ and $I_n^f \cap I_m^f = \varnothing$ for $n \neq m$. For $\alpha \subseteq \mathbb{N}$ we abbreviate $\bigcup\{I_n^f: n \in \alpha\}$ by I_α^f. The function f is called *polynomially honest* if f is recursive and there is a polynomial p such that $f(n)$ can be computed in $p(f(n))$ steps. Note that, for polynomially honest f with $f(n) > n$ and for $\alpha \in \mathscr{P}_{\mathbb{N}}$, $I_\alpha^f \in \mathscr{P}$. (Namely, given x compute the iterations $m_0 = f^0(0)$, $m_1 = f^1(0) = f(m_0)$,..., $m_n = f^n(0) = f(m_{n-1})$ of f up to the first n such that either $|x| < m_n$ or the computation of $m_n = f(m_{n-1})$ takes more than $p(|x|)$ steps, where p is the polynomial witnessing honesty of f. Then in either case $x \in I^f_n$, and $n \leqslant |x|$, since $f(m) > m$ for each m. So n can be found in polynomial time and it only remains to check whether $n \in \alpha$.) For $\alpha \subseteq \mathbb{N}$ and $i, k \in \mathbb{N}$ we let $k\alpha = \{kn : n \in \alpha\}$ and $k\alpha + i = \{kn + i: n \in \alpha\}$. Finally, a function g *dominates* f if $\forall n\ (f(n) < g(n))$. Note that every recursive function is dominated by some polynomially honest function.

3.1. LEMMA (join lemma). *Let C_0, C_1 be recursive sets and let $\mathscr{C}_0, \mathscr{C}_1$ be r.p. and c.f.v. classes such that $C_0 \cup C_1 \notin \mathscr{C}_0$ and $C_1 \notin \mathscr{C}_1$. Then there is a recursive function $g_0 : \mathbb{N} \to \mathbb{N}$ such that $g_0(n) > n$ for all n and the following holds. If g is a function which dominates g_0 and if α is an infinite and coinfinite set of natural numbers, then $(C_0 \cap I^g_\alpha) \cup C_1 \notin \mathscr{C}_0 \cup \mathscr{C}_1$.*

Proof. W.l.o.g. $\mathscr{C}_0$ and $\mathscr{C}_1$ are nonempty, say U_0, U_1 are recursive universal sets for $\mathscr{C}_0, \mathscr{C}_1$, respectively. Let

$$g_0(n) = \mu m > n\ (\forall k \leqslant n \exists x, y\ (n \leqslant |x|, |y| < m \text{ and } C_0 \cup C_1(x) \neq U_0^{(k)}(x)\ \&\ C_1(y) \neq U_1^{(k)}(y))).$$

Obviously g_0 is partial recursive and $g_0(n) > n$. Moreover, since $C_0 \cup C_1 \notin \mathscr{C}_0$, $C_1 \notin \mathscr{C}_1$, and $\mathscr{C}_0$ and $\mathscr{C}_1$ are closed under finite variants, for each k there are infinitely many x and y such that $C_0 \cup C_1(x) \neq U_0^{(k)}(x)$ and $C_1(y) \neq U_1^{(k)}(y)$. So g_0 is total.

Now fix g and α as in the premise of the lemma. We have to show $(C_0 \cap I^g_\alpha) \cup C_1 \notin \mathscr{C}_0$ and $(C_0 \cap I^g_\alpha) \cup C_1 \notin \mathscr{C}_1$. For a proof of the former we have to show $(C_0 \cap I^g_\alpha) \cup C_1 \neq U_0^{(k)}$ for each k. So fix k and by infinity of α choose $n \geqslant k$ with $n \in \alpha$. Then

$$\forall x \in \Sigma^*(g^n(0) \leqslant |x| < g^{n+1}(0) \to C_0 \cup C_1(x) = (C_0 \cap I^g_\alpha) \cup C_1(x)).$$

Moreover, by choice of g and by definition of g_0, $g^n(0) \geqslant n$, $g^n(0) < g_0(g^n(0)) \leqslant g^{n+1}(0)$ and there is some string x such that

$$g^n(0) \leqslant |x| < g_0(g^n(0)) \quad \text{and} \quad C_0 \cup C_1(x) \neq U_0^{(k)}(x).$$

All this implies $(C_0 \cap I^g_\alpha) \cup C_1(x) \neq U_0^{(k)}(x)$. The proof for the second claim is similar, now using that, by coinfinity of α, for each k there is some $n \geqslant k$ such that $n \notin \alpha$ and thus

$$\forall x \in \Sigma^*(g^n(0) \leqslant |x| < g^{n+1}(0) \to C_1(x) = (C_0 \cap I^g_\alpha) \cup C_1(x)).$$

To illustrate the way Lemma 3.1 can be used to obtain nontrivial suprema (joins) of polynomial degrees, we reprove Ladner's splitting theorem.

3.2. COROLLARY (Ladner, 1975). *For any recursive set $A \notin \mathscr{P}$ there is a set $B \in \mathscr{P}$ such that* $\deg_r^p(A \cap B) < \deg_r^p(A)$, $\deg_r^p(A \cap \bar{B}) < \deg_r^p(A)$ *and* $\deg_r^p(A) = \deg_r^p(A \cap B) \cup \deg_r^p(A \cap \bar{B})$. *Hence any nonzero p–r-degree is join reducible.*

Proof. Fix a recursive set $A \notin \mathscr{P}$. Note that, for any $B \in \mathscr{P}$, $A =_m^p (A \cap B) \oplus (A \cap \bar{B})$ and thus $\deg_r^p(A) = \deg_r^p(A \cap B) \cup \deg_r^p(A \cap \bar{B})$. Hence it suffices to find $B \in \mathscr{P}$ such that $A \not\leq_T^p A \cap B$ and $A \not\leq_T^p A \cap \bar{B}$.

Let g_0 be the recursive function obtained by the join lemma when applied to $C_0 = A$, $C_1 = \varnothing$, $\mathscr{C}_0 = \varnothing$, and $\mathscr{C}_1 = \deg_T^p(A)$. Then for any polynomially honest function g which dominates g_0, $I_{2\mathbb{N}}^g \in \mathscr{P}$, $\overline{I_{2\mathbb{N}}^g} = I_{2\mathbb{N}+1}^g$, $A \cap I_{2\mathbb{N}}^g \notin \mathscr{C}_0 \cup \mathscr{C}_1$, and $A \cap I_{2\mathbb{N}+1}^g \notin \mathscr{C}_0 \cup \mathscr{C}_1$. So $B = I_{2\mathbb{N}}^g$ has the desired properties.

As a second application of the join lemma we prove a variant of Breidbart's splitting theorem.

3.3. COROLLARY (Breidbart, 1978). *Let A be an infinite and coinfinite recursive set. Then there is a set $B \in \mathscr{P}$ such that $A \cap B$, $A \cap \bar{B}$, $\bar{A} \cap B$, and $\bar{A} \cap \bar{B}$ are infinite.*

Proof. We apply the join lemma twice, once to $C_0 = A$, $C_1 = \varnothing$, $\mathscr{C}_0 = \{D: D \text{ finite}\}$, $\mathscr{C}_1 = \varnothing$; and once to $C_0 = \bar{A}$, $C_1 = \varnothing$, $\mathscr{C}_0 = \{D: D \text{ finite}\}$, $\mathscr{C}_1 = \varnothing$; and we let g be any polynomially honest function dominating the resulting functions g_0. Then, as one can easily see, $B = I_{2\mathbb{N}}^g$ has the desired properties.

Note that Corollary 3.3 implies that, for any infinite recursive set A, $A \cap B$ and $A \cap \bar{B}$ are infinite for some $B \in \mathscr{P}$ (for cofinite A apply the corollary to $A \cap 0\Sigma^*$). Since $\mathscr{P}$ is closed under intersection, it follows that any infinite set $A \in \mathscr{P}$ possesses an infinite subset $A' \in \mathscr{P}$ such that the difference $A - A'$ is infinite. A direct proof for these facts can be found in Schöning (1982a).

We now turn to our second technical lemma.

3.4. LEMMA (meet lemma). *For any recursive set D there is a recursive function $g_1 : \mathbb{N} \to \mathbb{N}$, $g_1(n) > n$, such that the following holds. If g is a polynomially honest function which dominates g_1 and if $\alpha, \beta \in \mathscr{P}_{\mathbb{N}}$ and $E \subseteq \Sigma^*$ is recursive, then*

$$\deg_r^p((D \cap I_{2\alpha \cap 2\beta}^g) \oplus E) = \deg_r^p((D \cap I_{2\alpha}^g) \oplus E) \cap \deg_r^p((D \cap I_{2\beta}^g) \oplus E).$$

Proof. The proof is a straightforward variant of the proof of Landweber, Lipton, and Robertson's minimal pair theorem (1981, Theorem 14). Similar arguments have also been used by Chew and Machtey (1981) and Schöning (1984).

Given D, let g_1 be the step counting function of some deterministic Turing machine computing D such that $g_1(n) > n$. Now fix g, α, β, E as in the premise of the lemma. Then $I^g_{2\alpha}$, $I^g_{2\beta}$, $I^g_{2\alpha \cap 2\beta} \in \mathscr{P}$, and $I^g_{2\alpha} \cap I^g_{2\beta} = I^g_{2\alpha \cap 2\beta}$. Hence

$$(D \cap I^g_{2\alpha \cap 2\beta}) \oplus E \leqslant^p_m (D \cap I^g_{2\alpha}) \oplus E$$

and

$$(D \cap I^g_{2\alpha \cap 2\beta}) \oplus E \leqslant^p_m (D \cap I^g_{2\beta}) \oplus E.$$

So it suffices to show that, for any recursive set F satisfying

$$F \leqslant^p_r (D \cap I^g_{2\alpha}) \oplus E \qquad \text{and} \qquad F \leqslant^p_r (D \cap I^g_{2\beta}) \oplus E, \tag{3.1}$$

$F \leqslant^p_r (D \cap I^g_{2\alpha \cap 2\beta}) \oplus E$ holds. We do this for $r = T$ and leave the similar proofs for the other reducibilities to the reader.

Fix F recursive such that (3.1) holds for $r = T$. Then there are polynomial-time bounded deterministic oracle Turing machines M_1 and M_2 such that $F = M_1((D \cap I^g_{2\alpha}) \oplus E)$ and $F = M_2((D \cap I^g_{2\beta}) \oplus E)$. Fix a polynomial p which bounds the running times of M_1 and M_2. The following algorithm computes F from $(D \cap I^g_{2\alpha \cap 2\beta}) \oplus E$ in polynomial time.

Input x:
  Compute $n \in \mathbb{N}$ and $i \leqslant 1$ such that $0^{p(|x|)} \in I^g_{2n+i}$
  if $n \in \alpha$
    then simulate the computation $M_2((D \cap I^g_{2\beta}) \oplus E)(x)$ as follows:
        for query $0y$, $y \notin I^g_{2\beta}$, answer no
        for query $0y$, $y \in I^g_{2(\beta - \alpha)}$, compute $D(y)$
        for other queries use oracle $(D \cap I^g_{2\alpha \cap 2\beta}) \oplus E$
    else simulate the computation $M_1((D \cap I^g_{2\alpha}) \oplus E)(x)$ as follows:
        for query $0y$, $y \notin I^g_{2\alpha}$, answer no
        for query $0y$, $y \in I^g_{2(\alpha - \beta)}$, compute $D(y)$
        for other queries use oracle $(D \cap I^g_{2\alpha \cap 2\beta}) \oplus E$
  end if
end

Obviously the above algorithm correctly computes F. To show that it works in polynomial time, we have to prove that there is a polynomial q such that, for any input x and for any query $0y$ in the above computation of $F(x)$ which is answered by computing $D(y)$, $D(y)$ can be computed in $q(|x|)$ steps. So fix such x and $0y$. Then either

$$0^{p(|x|)} \in I^g_{2\alpha} \cup I^g_{2\alpha+1} \qquad \text{and} \qquad y \in I^g_{2(\beta - \alpha)}$$

or

$$0^{p(|x|)} \in I^g_{2\bar{\alpha}} \cup I^g_{2\bar{\alpha}+1} \qquad \text{and} \qquad y \in I^g_{2(\alpha-\beta)}.$$

Since there are only queries about strings of length $\leqslant p(|x|)$, this implies $t+2 \leqslant s$ for t and s such that $0^{p(|x|)} \in I^g_s$ and $y \in I^g_t$. So, by choice of g, $g_0(|y|) \leqslant g(|y|) < p(|x|)$, whence, by choice of g_0, $D(y)$ can be computed in less than $p(|x|)$ steps. This completes the proof of the lemma.

The meet lemma, in combination with the join lemma, can be applied to yield nontrivial meets in the polynomial time degrees. We conclude this section by giving an example.

3.5. COROLLARY (Landweber et al. (1981), Chew and Machtey (1981)). *Every nonzero polynomial time degree* **a** *bounds a minimal pair, i.e.,*

$$\forall \mathbf{a} > \mathbf{0}\ \exists \mathbf{b}_0, \mathbf{b}_1 (\mathbf{0} < \mathbf{b}_0, \mathbf{b}_1 \leqslant \mathbf{a}\ \&\ \mathbf{0} = \mathbf{b}_0 \cap \mathbf{b}_1).$$

Proof. Given $\mathbf{a} > \mathbf{0}$, fix $A \in \mathbf{a}$. Apply the Join Lemma to $C_0 = A$, $C_1 = \emptyset$, $\mathscr{C}_0 = \mathscr{P}$, $\mathscr{C}_1 = \emptyset$, and the meet lemma to $D = A$, and let g_0 and g_1, respectively, be the resulting functions. Now for any polynomially honest g which dominates g_0 and g_1 let $B_0 = I^g_{4\mathbb{N}} \cap A$, $B_1 = I^g_{4\mathbb{N}+2} \cap A$, $\mathbf{b}_0 = \deg^p(B_0)$, and $\mathbf{b}_1 = \deg^p(B_1)$. Obviously, $\mathbf{b}_0, \mathbf{b}_1 \leqslant \mathbf{a}$. Moreover, by the join lemma, $\mathbf{0} < \mathbf{b}_0, \mathbf{b}_1$. Finally, by the meet lemma applied to $\alpha = 2\mathbb{N}$, $\beta = 2\mathbb{N} + 1$, and $E = \emptyset$, $\mathbf{0} = \mathbf{b}_0 \cap \mathbf{b}_1$.

4. EMBEDDING THEOREMS

We first review some notions from lattice theory. A partially ordered (p.o.) set $\mathscr{L} = \langle L; \leqslant \rangle$ is a *lattice* if, for all $a, b \in L$, the supremum $a \cup b = \sup\{a, b\}$ (the *join* of a and b) and the infimum $a \cap b = \inf\{a, b\}$ (the *meet* of a and b) exist. If only joins exist then $\mathscr{L}$ is an *upper semilattice* (u.s.l.). The least (greatest) element of a p.o. set $\mathscr{L} = \langle L; \leqslant \rangle$ is denoted by $0_{\mathscr{L}}$ ($1_{\mathscr{L}}$) or simply by 0 (1). An element x of an (upper semi) lattice is *join* (*meet*) *reducible* if it is the join (meet) of two lesser (greater) elements. An *order embedding* of a p.o. set $\mathscr{L}_1 = \langle L_1; \leqslant_1 \rangle$ into a p.o. set $\mathscr{L}_2 = \langle L_2; \leqslant_2 \rangle$ is a one-to-one map $f: L_1 \to L_2$ such that

$$\forall a, b \in L_1\ (a \leqslant_1 b \to f(a) \leqslant_2 f(b)).$$

An order embedding of a lattice $\mathscr{L}_1$ into an upper semilattice $\mathscr{L}_2$ is a (*lattice*) *embedding* if

$$\forall a, b \in L_1\ (f(a \cup b) = f(a) \cup f(b), f(a) \cap f(b) \text{ exists and}$$
$$f(a \cap b) = f(a) \cap f(b)).$$

If $\mathscr{L}_1$ is embeddable in $\mathscr{L}_2$ then we also say $\mathscr{L}_1$ is a *sublattice* of $\mathscr{L}_2$ (up to isomorphism). We say an embedding f *preserves* the *least element* or 0 (*greatest element* or 1) if $f(0_{\mathscr{L}_1}) = 0_{\mathscr{L}_2}$ ($f(1_{\mathscr{L}_1}) = 1_{\mathscr{L}_2}$) or $0_{\mathscr{L}_1}(1_{\mathscr{L}_1})$ does not exist. A lattice $\mathscr{L} = \langle L; \leqslant \rangle$ is *distributive* if

$$\forall a, b, c \in L\ ((a \cup b) \cap (a \cup c) = a \cup (b \cap c)).$$

An upper semilattice $\mathscr{L}$ is *distributive* if

$$\forall a, b, c \in L\ (c \leqslant a \cup b \rightarrow \exists c_0 \leqslant a\ \exists c_1 \leqslant b\ (c = c_0 \cup c_1)).$$

Note that any sublattice of a distributive u.s.l. is distributive in the lattice sense. A distributive lattice $\mathscr{L}$ is *Boolean* if $\mathscr{L}$ possesses 0 and 1, $0 \neq 1$, and $\mathscr{L}$ is *complemented*, i.e., $\forall a \in L \exists \bar{a} \in L\ (a \cup \bar{a} = 1$ and $a \cap \bar{a} = 0)$. A Boolean lattice is *atomless* if $\forall a \in L - \{0\}\ \exists b \in L\ (0 < b < a)$. A proper subset $I \neq \varnothing$ of L is an *ideal* of the u.s.l. $\mathscr{L} = \langle L; \leqslant \rangle$ if I is closed under joins and $\forall x \in I, \forall y \in L\ (y \leqslant x \rightarrow y \in I)$. The quotient u.s.l. $\mathscr{L}/I = \langle L^*; \leqslant^* \rangle$ of $\mathscr{L}$ over the ideal I is defined by $L^* = \{[x]: x \in L\}$, where $[x] = \{z \cup y: y \in I$ and $\exists y' \in I\ (x = z \cup y')\}$, and $[x] \leqslant^* [y]$ if $x \leqslant y \cup z$ for some $z \in I$. Note that, for a Boolean lattice $\mathscr{L}$, $\mathscr{L}/I$ is a Boolean lattice too.

For a more detailed treatment of the above notions and for proofs of the basic lattice theoretic results applied below we refer the reader to Grätzer (1978).

We will now show that any countable distributive lattice can be embedded in any interval of polynomial degrees by maps which preserve 0 and 1, respectively. Since it is well known from lattice theory that any countable distributive lattice (with at least two elements) can be embedded into the (up to isomorphism unique) countably infinite atomless Boolean lattice by a map which preserves both 0 and 1, it suffices to embed this particular Boolean lattice. For this sake we first exhibit an efficiently computable representation of the countable atomless Boolean lattice.

Let $\langle \mathscr{P}^*_{\mathbb{N}}; \leqslant^* \rangle$ be the quotient lattice of $\langle \mathscr{P}_{\mathbb{N}}; \subseteq \rangle$ over the ideal of finite sets; i.e., the elements of $\mathscr{P}^*_{\mathbb{N}}$ are the classes $[\alpha] = \{\beta: \alpha =^* \beta\}$ $(\alpha \in \mathscr{P}_{\mathbb{N}})$ and $[\alpha] \leqslant^* [\beta]$ if $\alpha \subseteq^* \beta$.

4.1. LEMMA. (a) *$\langle \mathscr{P}_{\mathbb{N}}; \subseteq \rangle$ is a Boolean lattice.*

(b) *$\langle \mathscr{P}^*_{\mathbb{N}}; \leqslant^* \rangle$ is a countable atomless Boolean lattice.*

Proof. (a) Obviously $\mathscr{P}_{\mathbb{N}}$ is closed under $\cup$ (union) and $\cap$ (intersection). So $\mathscr{P}_{\mathbb{N}}$ is a field of sets and thus a distributive lattice. Moreover $\mathscr{P}_{\mathbb{N}}$ has least element $\varnothing$ and greatest element $\mathbb{N}$ and it is closed under (set) complementation. So $\mathscr{P}_{\mathbb{N}}$ is Boolean.

(b) Since the class $\mathscr{F}$ of finite subsets of N is an ideal of $\mathscr{P}_{\mathbb{N}}$, the quotient lattice $\langle \mathscr{P}^*_{\mathbb{N}}; \leqslant^* \rangle$ of the Boolean lattice $\langle \mathscr{P}_{\mathbb{N}}; \subseteq \rangle$ over $\mathscr{F}$ is a Boolean lattice too. Obviously $\mathscr{P}^*_{\mathbb{N}}$ is countable. So it only remains to show that $\mathscr{P}^*_{\mathbb{N}}$ has no atoms, i.e., that any infinite set $A \in \mathscr{P}_{\mathbb{N}}$ has a subset $B \in \mathscr{P}_{\mathbb{N}}$ such that both B and $A - B$ are infinite. As pointed out in the previous section this is an immediate consequence of Breidbart's splitting theorem (Corollary 3.3) using the canonical embedding of $\mathscr{P}_{\mathbb{N}}$ into $\mathscr{P}$.

4.2. THEOREM (embedding theorem). *Let A and B be recursive sets such that $B <^p_r A$ and let $\mathscr{C}$ and $\mathscr{D}$ be r.p. and c.f.v. classes of recursive sets such that $A \oplus B \notin \mathscr{C}$ and $\varnothing \oplus B \notin \mathscr{D}$. Then there is a polynomially honest function g, $g(n) > n$, such that the following holds. The functions $f_i: \mathscr{P}_{\mathbb{N}} \to \Sigma^*$ and $f_i^*: \mathscr{P}^*_{\mathbb{N}} \to \mathbf{R}^p_r$ $(i = 0, 1)$ defined by*

$$f_0(\alpha) = (A \cap I^g_{2\alpha}) \oplus B \tag{4.1}$$

$$f_1(\alpha) = (A \cap I^g_{2\alpha \cup 2\mathbb{N}+1}) \oplus B \tag{4.2}$$

$$f_i^*([\alpha]) = \deg^p_r(f_i(\alpha)) \tag{4.3}$$

have the following properties:

If $\alpha \in \mathscr{P}_{\mathbb{N}}$ is infinte and $\beta \in \mathscr{P}_{\mathbb{N}}$ is coinfinite then $f_0(\alpha)$, $f_1(\beta) \notin \mathscr{C} \cup \mathscr{D} \cup \deg^p_r(A) \cup \deg^p_r(B)$. (4.4)

The function $f_0^*(f_1^*)$ gives an embedding of the atomless Boolean lattice $\langle \mathscr{P}^*_{\mathbb{N}}; \leqslant^* \rangle$ into the interval $[\deg^p_r(B), \deg^p_r(A)]$ which preserves the least (greatest) element. Moreover, for any $[\alpha] \in \mathscr{P}^*_{\mathbb{N}}$, there is a set $C \in f_i^*([\alpha])$ s.t. $C \leqslant^p_m A \oplus B$. (4.5)

Proof. Let $\mathscr{C}_0 = \mathscr{C} \cup \{E: E \leqslant^p_r B\}$ and $\mathscr{C}_1 = \mathscr{D} \cup \deg^p_r(A)$, and let g_0 and g_1 be the functions supplied by the join lemma and the meet lemma when applied to $C_0 = A \oplus \varnothing$, $C_1 = \varnothing \oplus B$, $\mathscr{C}_0, \mathscr{C}_1$, and $D = A$, respectively. Finally let g be any polynomially honest function which dominates both g_0 and g_1.

Then (4.4.) is immediate by the join lemma. For a proof of (4.5) we first recall some properties of the sets I^g_α observed in Section 3: For α, $\beta \subseteq \mathbb{N}$,

$$\alpha \subseteq \beta \leftrightarrow I^g_\alpha \subseteq I^g_\beta \tag{4.6}$$

$$\alpha \text{ finite} \leftrightarrow I^g_\alpha \text{ finite} \tag{4.7}$$

$$\alpha \in \mathscr{P}_{\mathbb{N}} \to I^g_\alpha \in \mathscr{P}. \tag{4.8}$$

Also note that for any recursive set E and for sets $F_0, F_1 \in \mathscr{P}$,

$$\deg_m^p(E \cap (F_0 \cup F_1)) = \deg_m^p(E \cap F_0) \cup \deg_m^p(E \cap F_1)$$

and $F_0 \subseteq F_1$ implies $E \cap F_0 \leqslant_m^p E \cap F_1$. So, by definition of f_i and by (4.6) and (4.8),

$$\deg_m^p(f_i(\alpha)) \cup \deg_m^p(f_i(\beta)) = \deg_m^p(f_i(\alpha \cup \beta)) \tag{4.9}$$

and

$$\alpha \subseteq \beta \to B \leqslant_m^p f_i(\alpha) \leqslant_m^p f_i(\beta) \leqslant_m^p A \oplus B \tag{4.10}$$

for $\alpha, \beta \in \mathscr{P}_{\mathbb{N}}$. Moreover, by (4.7) and closure of p–m-degrees under finite variants,

$$\alpha =^* \beta \to f_i(\alpha) =_m^p f_i(\beta). \tag{4.11}$$

The last fact implies that the functions f_i^* are well defined. By (4.10), the range of f_i^* is contained in $[\deg_r^p(B), \deg_r^p(A)]$ and the appendix of (4.5) holds. Furthermore, since $f_0(\varnothing) = \varnothing \oplus B$ and $f_1(\mathbb{N}) = A \oplus B$, $f_0^*(0) = \deg_r^p(B)$ and $f_1^*(1) = \deg_r^p(A)$. So it only remains to show that f_i^* embeds $\langle \mathscr{P}_{\mathbb{N}}^*; \leqslant^* \rangle$ into $\langle \mathbf{R}_r^p; \leqslant \rangle$. For a proof of the latter it suffices to show for $\alpha, \beta \in \mathscr{P}_{\mathbb{N}}$,

$$[\alpha] \leqslant^* [\beta] \leftrightarrow f_i^*([\alpha]) \leqslant f_i^*([\beta])$$

$$f_i^*([\alpha \cup \beta]) = f_i^*([\alpha]) \cup f_i^*([\beta])$$

$$f_i^*([\alpha \cap \beta]) = f_i^*([\alpha]) \cap f_i^*([\beta]).$$

By definition of f_i^* (and by (4.11)) we may replace these conditions by

$$\alpha \subseteq^* \beta \leftrightarrow f_i(\alpha) \leqslant_r^p f_i(\beta) \tag{4.12}$$

$$\deg_r^p(f_i(\alpha \cup \beta)) = \deg_r^p(f_i(\alpha)) \cup \deg_r^p(f_i(\beta)) \tag{4.13}$$

$$\deg_r^p(f_i(\alpha \cap \beta)) = \deg_r^p(f_i(\alpha)) \cap \deg_r^p(f_i(\beta)). \tag{4.14}$$

Now (4.13) and the direction "$\rightarrow$" in (4.12) are immediate by (4.9) and (4.10), respectively. Moreover, (4.14) holds by the meet lemma and definition of f_i. (Note that $f_1(\alpha) =_m^p (A \cap I_{2\alpha}^g) \oplus ((A \cap I_{2\mathbb{N}+1}^g) \oplus B)$.) It only remains to prove the other direction of (4.12). So fix $\alpha, \beta \in \mathscr{P}_{\mathbb{N}}$ such that $f_i(\alpha) \leqslant_r^p f_i(\beta)$ and, for a contradiction, assume $\gamma = \alpha - \beta$ is infinite. Note that $\gamma \in \mathscr{P}_{\mathbb{N}}$. So, by (4.10), $f_i(\gamma) \leqslant_m^p f_i(\alpha)$ and thus, by choice of α and β, $f_i(\gamma) \leqslant_r^p f_i(\beta)$. Since trivially $f_i(\gamma) \leqslant_r^p f_i(\gamma)$, this and (4.14) imply

$f_i(\gamma) \leqslant_r^p f_i(\beta \cap \gamma) = f_i(\varnothing)$, i.e., by (4.10), $f_i(\gamma) =_r^p f_i(\varnothing)$. So, for $i=0$, $f_0(\gamma) \in \deg_r^p(B)$ contrary to (4.4). Moreover, by (4.13) and (4.10),

$$\begin{aligned} \deg_r^p(f_i(\mathbb{N})) &= \deg_r^p(f_i(\gamma \cup \bar{\gamma})) = \deg_r^p(f_i(\gamma)) \cup \deg_r^p(f_i(\bar{\gamma})) \\ &= \deg_r^p(f_i(\varnothing)) \cup \deg_r^p(f_i(\bar{\gamma})) = \deg_r^p(f_i(\bar{\gamma})), \end{aligned}$$

whence, for $i=1$, $f_1(\bar{\gamma}) \in \deg_r^p(A)$ contrary to (4.4). This completes the proof of Theorem 4.2.

4.3. COROLLARY. *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{R}_r^p$ *be given such that* $\mathbf{b} < \mathbf{a}$ *and let* $\mathscr{L}$ *be a countable distributive lattice. Then there are embeddings of* $\mathscr{L}$ *into the interval* $[\mathbf{b}, \mathbf{a}]$ *which preserve the least and greatest element, respectively. If moreover* $\mathbf{a}, \mathbf{b} \in \mathbf{NP}_r^p$ *then the embeddings can be chosen to be into* $[\mathbf{b}, \mathbf{a}] \cap \mathbf{NP}_r^p$.

Proof. Fix $A \in \mathbf{a}$ and $B \in \mathbf{b}$ (with $A, B \in \mathscr{NP}$ if $\mathbf{a}, \mathbf{b} \in \mathbf{NP}_r^p$) and let $\mathscr{C} = \mathscr{D} = \varnothing$. Then Theorem 4.2 yields embeddings f_i^* of the countable atomless Boolean lattice $\langle \mathscr{P}_{\mathbb{N}}^*; \leqslant^* \rangle$ into the interval $[\mathbf{b}, \mathbf{a}]$ which preserve i, $i=0, 1$. Moreover, by downward closure of $\mathscr{NP}$ under $\leqslant_m^p$ and by the appendix of (4.5), for $A, B \in \mathscr{NP}$ and thus $A \oplus B \in \mathscr{NP}$, range$(f_i^*) \subseteq \mathbf{NP}_r^p$. So the claim follows from the embedding universality for countable distributive lattices of the countable atomless Boolean lattice.

Since any countable p.o. set can be order embedded in the countable atomless Boolean lattice, Corollary 4.3 implies the following result on partial suborderings of the polynomial degrees which has been stated in Mehlhorn (1976) without proof.

4.4. COROLLARY (Mehlhorn, 1976). *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{R}_r^p$ $(\mathbf{NP}_r^p)$ *be given such that* $\mathbf{b} < \mathbf{a}$. *Any countable p.o. set can be order embedded in* $[\mathbf{b}, \mathbf{a}](\cap \mathbf{NP}_r^p)$.

We can extend Corollary 4.3 by showing that the embeddings may be chosen to be incomparable with any given finite (in fact any r.p.) class of p–r-degrees which avoid the upper and lower cones of **a** and **b**, respectively.

4.5. COROLLARY (main embedding theorem). *Let* $\mathbf{a}, \mathbf{b} \in \mathbf{R}_r^p$ $(\mathbf{NP}_r^p)$ *be given such that* $\mathbf{b} < \mathbf{a}$ *and let* $\mathbf{C}$ *be an r.p. class of* p–r-*degrees such that*

$$\forall \mathbf{c} \in \mathbf{C}\ (\mathbf{c} \nleq \mathbf{b} \textit{ and } \mathbf{a} \nleq \mathbf{c}).$$

For any countable distributive lattice $\mathscr{L}$ *there are embeddings* F_0 *and* F_1 *of* $\mathscr{L}$ *into* $[b, a]$ *which preserve* 0 *and* 1, *respectively, and such that*

$$\forall \mathbf{c} \in \mathbf{C}, \forall a \in L - \{i\}\ (\mathbf{c} \nleq F_i(a)\ \&\ F_i(a) \nleq \mathbf{c}) \qquad (i=0, 1) \tag{4.15}$$

(*and range*$(F_i) \subseteq \mathbf{NP}_r^p$, $i=0, 1$).

Proof. Like the proof of Corollary 4.3 with $\mathscr{C} = [\{B\}, \{C\colon \deg_r^p(C) \in \mathbf{C}\}]_r$ and $\mathscr{D} = [\{C\colon \deg_r^p(C) \in \mathbf{C}\}, \{A\}]_r$. Then (4.4) ensures (4.15) for $\mathscr{L} = \langle \mathscr{P}_{\mathbb{N}}^*; \leqslant^* \rangle$.

Corollary 4.5 unifies and extends most of the previous results on the structure of the polynomial time degrees of recursive sets. Moreover, assuming $\mathscr{P} \neq \mathscr{N}\mathscr{P}$, all results on the polynomial degrees of $\mathscr{N}\mathscr{P}$-sets proved in Balcazar and Diaz (1982), Chew and Machtey (1981), Ladner (1973, 1975), Landweber, Lipton, and Robertson (1981), Mehlhorn (1976), and Schmidt (1984) are direct consequences of Corollary 4.5 and the existence of $\mathscr{N}\mathscr{P}$-complete problems.

We obtain these results by appropriate choices of the lattice $\mathscr{L}$. In the following we give two examples.

First let $\mathscr{L}$ be the 3-element total ordering. Then Corollary 4.5 gives density of the polynomial time degrees (Ladner (1975)), i.e.,

$$\forall \mathbf{b} < \mathbf{a}\ \exists \mathbf{d} \qquad (\mathbf{b} < \mathbf{d} < \mathbf{a}).$$

Moreover, $\mathbf{d}$ can be chosen incomparable with any finite (or recursively presentable) class in the interval $(\mathbf{b}, \mathbf{a})$. (This extends the main theorem of Balcazar and Diaz (1982).) Hence no interval of polynomial degrees is totally ordered (Ladner, 1975) and no finite (r.p.) anti-chain in a given interval is maximal.

Now let $\mathscr{L}$ be the 2-atom Boolean lattice $\mathscr{B}_2$. Then the 1-preserving embeddings of $\mathscr{B}_2$ give Ladner's splitting theorem, i.e., the fact that every nonzero degree $\mathbf{a}$ is join-reducible. In fact the splitting can be done above any lesser degree $\mathbf{b}$ (Ladner, 1975) and it can be chosen to be incomparable with any intermediate degree $\mathbf{c}$, i.e.,

$$\forall \mathbf{b} < \mathbf{c} < \mathbf{a}\ \exists \mathbf{d}_0, \mathbf{d}_1 \qquad (\mathbf{b} < \mathbf{d}_0, \mathbf{d}_1 < \mathbf{a}\ \&\ \mathbf{a} = \mathbf{d}_0 \cup \mathbf{d}_1\ \&\ \mathbf{d}_0 \mid \mathbf{c}\ \&\ \mathbf{d}_1 \mid \mathbf{c}).$$

Taking 0-preserving embeddings of $\mathscr{B}_2$ we obtain the dual result

$$\forall \mathbf{b} < \mathbf{c} < \mathbf{a}\ \exists \mathbf{d}_0, \mathbf{d}_1 \qquad (\mathbf{b} < \mathbf{d}_0, \mathbf{d}_1 < \mathbf{a}\ \&\ \mathbf{b} = \mathbf{d}_0 \cap \mathbf{d}_1\ \&\ \mathbf{d}_0 \mid \mathbf{c}\ \&\ \mathbf{d}_1 \mid \mathbf{c}).$$

This shows that every degree is meet-reducible (Landweber *et al.*, 1981) and, by taking $\mathbf{b} = \mathbf{0}$, that every degree bounds a minimal pair (Landweber *et al.*, 1981 and Chew and Machtey, 1981). Moreover, if we let $\mathbf{a} = \mathbf{0}'$, $\mathbf{0}'$ the degree of NP-complete problems, and if we assume that $\mathscr{P} \neq \mathscr{N}\mathscr{P}$ then, for each intermediate NP-degree $\mathbf{c}$, we obtain a minimal pair of degrees of NP-sets incomparable with $\mathbf{c}$. So the class of minimal pairs of NP-degrees is unbounded in $\mathbf{NP} - \{\mathbf{0}'\}$.

It is natural to ask the following questions on possible extensions of our embedding theorem: (1) Can the embeddings be chosen to preserve both the least and the greatest elements? (2) Are there any *nondistributive* lattices which can be embedded in the polynomial time degrees?

In Ambos–Spies (to appear) we negatively answer the first question for p–m-degrees. There is a nonzero recursive p–m-degree $\mathbf{a}$ which is not supremum of a minimal pair. So there is no embedding of the two atom Boolean lattice into the interval $[\mathbf{0}, \mathbf{a}]$ which preserves 0 and 1. It follows that no *somewhere complemented* lattice can be embedded into the interval $[\mathbf{0}, \mathbf{a}]$ by a map which preserves 0 and 1, i.e., no lattice $\mathscr{L}$ containing elements $x, y \notin \{0, 1\}$ such that $0 = x \cap y$ and $x \cup y = 1$ has such an embedding. On the other hand, in Ambos–Spies (1986) we show that every finite distributive lattice which is nowhere complemented can be embedded in every interval of polynomial time degrees by maps which preserve both the least and the greatest elements. The answer to the second question depends on the underlying reducibility notion. In Ambos–Spies (1984) we have shown that the upper semilattices of the polynomial time degrees of recursive sets with respect to p–btt, p–tt, p–d, and p–T reducibilities are nondistributive (in the u.s.l. sense) and as we shall show elsewhere these structures, as well as the p–c-degrees, indeed possess nondistributive sublattices. The structures of p–m– and p–1–tt-degrees, however, are distributive.

4.6. Lemma (Ambos–Spies, 1984). *The u.s.l. of p–m (p–1–tt) degrees is distributive.*

Proof. Fix $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^p_m$ such that $\mathbf{a} \leqslant \mathbf{b} \cup \mathbf{c}$. We will show that there are degrees $\mathbf{d}, \mathbf{e} \in \mathbf{R}^p_m$ such that $\mathbf{d} \leqslant \mathbf{b}$, $\mathbf{e} \leqslant \mathbf{c}$, and $\mathbf{a} = \mathbf{d} \cup \mathbf{e}$. Choose recursive sets $A \in \mathbf{a}$, $B \in \mathbf{b}$, and $C \in \mathbf{c}$. Then $A \leqslant^p_m B \oplus C$, say via f. Now let $F = \{x : f(x) \in 0\Sigma^*\}$, and set $D = A \cap F$, $E = A \cap \bar{F}$, $\mathbf{d} = \deg^p_m(D)$, and $\mathbf{e} = \deg^p_m(E)$. Obviously $F \in \mathscr{P}$. Hence $A =^p_m D \oplus E$, i.e., $\mathbf{a} = \mathbf{d} \cup \mathbf{e}$. Moreover, $D \leqslant^p_m B \oplus \varnothing$ via f and $E \leqslant^p_m \varnothing \oplus C$ via f, whence $\mathbf{d} \leqslant \mathbf{b}$ and $\mathbf{e} \leqslant \mathbf{c}$.

The proof for p–1–tt-degrees is similar.

The above results give a complete characterisation of the sublattices of $\mathbf{R}^p_m$ ($\mathbf{R}^p_{1\text{-}tt}$).

4.7. Corollary. *Let $\mathscr{L}$ be a lattice. The following are equivalent.*

(a) *$\mathscr{L}$ is countable and distributive.*

(b) *$\mathscr{L}$ can be embedded in $\mathbf{R}^p_m(\mathbf{R}^p_{1\text{-}tt})$.*

(c) *$\mathscr{L}$ can be embedded in any interval of $\mathbf{R}^p_m$ ($\mathbf{R}^p_{1\text{-}tt}$) by a map which preserves the least element.*

(d) *$\mathscr{L}$ can be embedded in any interval of $\mathbf{R}^p_m$ ($\mathbf{R}^p_{1\text{-}tt}$) by a map which preserves the greatest element.*

(e) *(Assuming $\mathscr{P} \neq \mathscr{NP}$) $\mathscr{L}$ can be embedded in $\mathbf{NP}^p_m$ ($\mathbf{NP}^p_{1\text{-}tt}$).*

We conclude this section with an application of our embedding results to the structure of the p–r_2-degrees contained in a single p–r_1-degree, where r_2 is stronger than r_1 ($r_1, r_2 \in \{m, 1\text{–tt}, \text{btt}, \text{tt}, c, d, T\}$).

4.8. COROLLARY. *Let r_1, r_2 be polynomial time reducibilities such that r_2 is stronger than r_1, and let A be a recursive set. Then either* $\deg^p_{r_1}(A)$ *consists of a single p–r_2-degree or* $\deg^p_{r_1}(A)$ *consists of infinitely many p–r_2-degrees. In fact in the latter case any countable distributive lattice (p.o. set) can be (order) embedded in* $\langle \{\deg^p_{r_2}(B) : B =^p_{r_1} A\}; \leqslant \rangle$.

Proof. If $\deg^p_{r_1}(A)$ contains two non–p–r_2-equivalent elements, say B and C, where w.l.o.g. $B \not\leqslant^p_{r_2} C$, then $C <^p_{r_2} B \oplus C$ and $[\deg^p_{r_2}(C), \deg^p_{r_2}(B \oplus C)]$ is contained in $\deg^p_{r_1}(A)$. So the claim follows from Corollaries 4.3 and 4.4.

A similar argument shows

4.9. COROLLARY. *Let r_1 and r_2 be polynomial time reducibilities such that r_2 is stronger than r_1. If there is a set which is $\mathcal{NP}$-complete with respect to r_1 but which is not $\mathcal{NP}$-complete with respect to r_2 then* $\mathbf{NPC}^p_{r_1,r_2} = \{\deg^p_{r_2}(A) : A$ *is $\mathcal{NP}$-complete w.r.t. r_1*$\}$ *is infinite. In fact any countable distributive lattice (p.o. set) can be (order) embedded in the u.s.l.* $\mathbf{NPC}^p_{r_1,r_2}$ *by a map which preserves the greatest element.*

5. APPLICATIONS

In this final section we use the embedding theorem to gain some insight in the distribution of the polynomial time degrees of sets with certain interesting structural properties. We will consider three examples: non-selfdual sets, non-mitotic sets, and (assuming $\mathcal{NP} \neq co\mathcal{NP}$) sets which can be p–T-reduced to some $\mathcal{NP}$-set but which are themselves neither in $\mathcal{NP}$ nor in $co\mathcal{NP}$. The classes of these sets (for the last example an appropriate superclass) share the following property.

5.1. DEFINITION. A nonempty class $\mathcal{B}$ of recursive sets is *normal* if

(i) the complement $\bar{\mathcal{B}}$ of $\mathcal{B}$ (relative to the class of recursive sets) is locally recursively presentable and closed under finite variants and

(ii) $A \in \mathcal{B}$ implies $\emptyset \oplus A \in \mathcal{B}$ and $A \oplus \emptyset \in \mathcal{B}$.

The following theorem gives some information on the structure of the polynomial time degrees of normal classes.

5.2. THEOREM. *Let $\mathscr{B}$ be normal and let* $\mathbf{B}$ *be the class of p–r-degrees of sets in* $\mathscr{B}$. *Moreover, let* $\mathscr{L} = \langle L; \leqslant \rangle$ *be any countable distributive lattice, let* $\mathbf{a}$, $\mathbf{b}$ *be p–r-degrees such that* $\mathbf{b} < \mathbf{a}$ *and let* $\mathbf{C}$ *be an r.p. class of p–r-degrees such that* $\forall \mathbf{c} \in \mathbf{C}$ $(\mathbf{c} \nleqslant \mathbf{b}$ *and* $\mathbf{a} \nleqslant \mathbf{c})$.

(i) *If* $\mathbf{b} \in \mathbf{B}$ *then there is an embedding f of* $\mathscr{L}$ *into* $[\mathbf{b}, \mathbf{a}]$ *such that f preserves* 0, range$(f) \subseteq \mathbf{B}$ *and*

$$\forall x \in L - \{0\}, \forall \mathbf{c} \in \mathbf{C} \qquad (f(x) \nleqslant \mathbf{c} \;\&\; \mathbf{c} \nleqslant f(x)).$$

(ii) *If* $\mathbf{a} \in \mathbf{B}$ *then there are embeddings* f_i *of* $\mathscr{L}$ *into* $[\mathbf{0}, \mathbf{a}]$ $(i = 0, 1)$ *such that* f_i *preserves i,* range$(f_i) \subseteq \mathbf{B} \cup \{\mathbf{0}\}$ *and*

$$\forall x \in L - \{i\}, \forall \mathbf{c} \in \mathbf{C} \qquad (f_i(x) \nleqslant \mathbf{c} \text{ and } \mathbf{c} \nleqslant f_i(x)).$$

Proof. W.l.o.g. it suffices to consider the lattice $\mathscr{L} = \langle \mathscr{P}^*_{\mathbb{N}}; \leqslant^* \rangle$. Then (i) and (ii) follow from the embedding theorem for the following choices of A, B, $\mathscr{C}$, and $\mathscr{D}$: For (i) let A be any element of $\mathbf{a}$, B be an element of $\mathbf{b}$ such that $B \in \mathscr{B}$, $\mathscr{C} = [\mathscr{S}(\mathbf{C}), \{A\}]_r$, $\mathscr{D} = [\{B\}, \mathscr{S}(\mathbf{C})]_r \cup (\{D: D \leqslant^p_r A\} \cap \bar{\mathscr{B}})$; for (ii) let A be an element of $\mathbf{a}$ such that $A \in \mathscr{B}$, $B = \emptyset$, $\mathscr{C} = [\mathscr{S}(\mathbf{C}), \{A\}]_r \cup (\{D: D \leqslant^p_r A\} \cap \bar{\mathscr{B}})$, $\mathscr{D} = [\{\emptyset\}, \mathscr{S}(\mathbf{C})]_r$, where $\mathscr{S}(\mathbf{C}) = \{C: \deg^p_r(C) \in \mathbf{C}\}$.

5.3. COROLLARY. *Let* $\mathscr{B}$ *be normal and let* $\mathbf{B}$ *be the class of the p–r-degrees of the members of* $\mathscr{B}$:

(i) $\mathbf{B}$ *is unbounded; in fact* $\forall \mathbf{c} \in \mathbf{R}^p_r - \{\mathbf{0}\}\ \exists \mathbf{d} \in \mathbf{B}$ $(\mathbf{d} \nleqslant \mathbf{c}$ *and* $\mathbf{c} \nleqslant \mathbf{d})$.

(ii) $\mathbf{B}$ *is dense, i.e.,* $\forall \mathbf{a}, \mathbf{b} \in \mathbf{B}$ $(\mathbf{a} < \mathbf{b} \rightarrow \exists \mathbf{c} \in \mathbf{B}$ $(\mathbf{a} < \mathbf{c} < \mathbf{b}))$.

(iii) *Any element* $\mathbf{a} \in \mathbf{B}$ *is meet reducible in* $\mathbf{B}$; *in fact*

$$\forall \mathbf{a} \in \mathbf{B}, \forall \mathbf{b} > \mathbf{a}, \exists \mathbf{c}_0, \mathbf{c}_1 \in \mathbf{B} \qquad (\mathbf{a} < \mathbf{c}_0, \mathbf{c}_1 < \mathbf{b} \;\&\; \mathbf{a} = \mathbf{c}_0 \cap \mathbf{c}_1).$$

(iv) *Any element* $\mathbf{a} \in \mathbf{B} - \{\mathbf{0}\}$ *is join reducible in* $\mathbf{B}$.

(v) $\mathbf{B} - \{\mathbf{0}\}$ *has neither minimal nor maximal elements.*

(vi) *Any nonzero element of* $\mathbf{B}$ *bounds a minimal pair of p–r-degrees in* $\mathbf{B}$.

Proof. Claims (ii), (iii), (iv), and (vi) are immediate by Theorem 5.2. (v) is a direct consequence of (iii) and (iv). For a proof of (i), fix $\mathbf{c} > \mathbf{0}$ and, by nonemptiness of $\mathscr{B}$, choose $\mathbf{b} \in \mathbf{B}$. W.l.o.g. $\mathbf{b} \leqslant \mathbf{c}$ or $\mathbf{c} \leqslant \mathbf{b}$. Moreover, by (iii), we may assume $\mathbf{b} \neq \mathbf{c}$. Now if $\mathbf{b} < \mathbf{c}$ then an application of part (i) of Theorem 5.2 to $\mathbf{b}$, $\mathbf{C} = \{\mathbf{c}\}$ and $\mathscr{L}$ a lattice containing a chain of length 2 yields a degree $\mathbf{d} \in \mathbf{B}$ incomparable with $\mathbf{c}$. If $\mathbf{c} < \mathbf{b}$ then an application of part (ii) to $\mathbf{a} = \mathbf{b}$ and $\mathbf{C}$ and $\mathscr{L}$ as above yields the desired degree $\mathbf{d}$.

Note that Corollary 5.3 subsumes a local version: For any degree $\mathbf{a} > \mathbf{0}$ which bounds an element of $\mathbf{B}$, the class $\mathbf{B}(<\mathbf{a}) = \{\mathbf{b}: \mathbf{b} \in \mathbf{B} \;\&\; \mathbf{b} < \mathbf{a}\}$ satisfies (ii)–(vi) in the restricted universe $\mathbf{R}^p_r(<\mathbf{a})$.

We now give some examples of normal classes.

Selfdual Sets

The p–many–one reducibility can be distinguished from the p–l–tt and weaker polynomial time reducibilities by the fact that in general a set cannot be p–m-reduced to its complement (Ladner, Lynch, and Selman, 1975). A set A for which $A \leqslant^p_m \bar{A}$ holds (or $A = \emptyset$ or $A = \Sigma^*$) is called *selfdual*; otherwise A is *non-selfdual*. Note that for any A, $A \oplus \bar{A}$ is selfdual. So any polynomial l–tt-degree contains a selfdual set. Moreover any polynomial time computable set is selfdual. Also note that for selfdual A and $B =^p_m A$, B and $\bar{B}$ are selfdual too. So in particular the class of selfdual sets is closed under finite variants and, for non-selfdual A, $A \oplus \emptyset$ and $\emptyset \oplus A$ fail to be selfdual too.

5.4. THEOREM. *The class of non-selfdual recursive sets is normal.*

Proof. Since Ladner *et al.* (1975) have shown that non-selfdual sets exist, by the above remarks it suffices to show that, for any recursive set B, the class $\mathscr{SD}(\leqslant^p_m B) = \{A : A$ selfdual and $A \leqslant^p_m B\}$ is r.p. The following inductively defined recursive set U is universal for $\mathscr{SD}(\leqslant^p_m B)$: For $m, n \in \mathbb{N}$ and $x \in \Sigma^*$ let $U^{(\langle m,n \rangle)}(x) = B(f_m(x))$ if

$$\forall y < x \qquad (f_n(y) < x \rightarrow U^{(\langle m,n\rangle)}(y) \neq U^{(\langle m,n\rangle)}(f_n(y)))$$

and $U^{(\langle m,n\rangle)}(x) = 0$ otherwise. Note that $U^{(\langle m,n\rangle)} = A$ if $A \leqslant^p_m B$ via f_m and $A \leqslant^p_m \bar{A}$ via f_n; otherwise $U(\langle m,n\rangle)$ is finite.

P-mitotic Sets

A set A is called *p–r-mitotic* if, for some $B \in \mathscr{P}$, $A =^p_r A \cap B =^p_r A \cap \bar{B}$ (in case of $r = m$, $\emptyset$ and Σ^* are assumed to be mitotic too). Intuitively speaking, p-mitotic sets can be efficiently split into two parts, both of the same complexity as the whole set. Thus p-mitoticity expresses some kind of redundancies in a given problem. It seems that all "natural" sets have this property. E.g., any p-cylinder is p-mitotic and, for any set A, $A \oplus A$ is p-mitotic (cf. Ambos-Spies, 1984a). So any polynomial time degree contains a p-mitotic set. For a detailed study of p–mitotic sets we refer the reader to Ambos–Spies (1984a). There we have shown that non–p–T-mitotic sets exist (and thus non–p–r-mitotic sets for the other reducibility notions r). We also show there, however, that there is a nonzero p–m-degree which consists only of p–m-mitotic sets.

5.5. THEOREM. *The class $\bar{\mathscr{M}}_r$ of non–p–r-mitotic recursive sets is normal.*

Proof. As remarked above $\bar{\mathscr{M}}_r$ is nonempty, and obviously $A \in \bar{\mathscr{M}}_r$ implies $A \oplus \varnothing \in \bar{\mathscr{M}}_r$ and $\varnothing \oplus A \in \bar{\mathscr{M}}_r$, and $\mathscr{M}_r$ is closed under finite variants. So it remains to show that for any recursive set A, $\mathscr{M}_r(\leqslant^p_m A)$ is recursively presentable. We show this for $r = m$ and leave the proofs for the other reducibilities to the reader. Recall that $A \cap B \leqslant^p_m A$ for $B \in \mathscr{P}$. So $A =^p_m A \cap B =^p_m A \cap \bar{B}$ iff $A \leqslant^p_m A \cap B$ and $A \leqslant^p_m A \cap \bar{B}$. The following inductively defined set U is universal for $\mathscr{M}_m(\leqslant^p_m A)$. For $i, k, m, n \in \mathbb{N}$, $x \in \Sigma^*$, and $s = \langle i, k, m, n \rangle$ we let $U^{(s)}(x) = A(f_i(x))$ if

$$\forall y < x \qquad [(f_k(y) < x \rightarrow U^{(s)}(y) = (U^{(s)} \cap P_n)(f_k(y))) \text{ and}$$
$$(f_m(y) < x \rightarrow U^{(s)}(y) = (U^{(s)} \cap \bar{P}_n)(f_m(y)))]$$

and $U^{(s)}(x) = 0$ otherwise. Note that either $U^{(s)} = f_i^{-1}(A)$, $U^{(s)} = f_k^{-1}(U^{(s)} \cap P_n)$, and $U^{(s)} = f_m^{-1}(U^{(s)} \cap \bar{P}_n)$ or $U^{(s)}$ is finite.

In a similar way one can show that for the variants of mitoticity introduced in Ambos–Spies (1984a) the classes of non-mitotic sets are normal too.

Δ^p_2-Sets

While any set p–m-reducible to a set in $\mathscr{NP}$ ($co\mathscr{NP}$) is in $\mathscr{NP}$ ($co\mathscr{NP}$) too, it is an open problem whether any set p–T-reducible to an $\mathscr{NP}$-set is again in $\mathscr{NP}$. The class of the latter sets is denoted by Δ^p_2,

$$\Delta^p_2 = \{A: \exists B \in \mathscr{NP}\ (A \leqslant^p_T B)\} = \{A: A \leqslant^p_T C\},$$

C some $\mathscr{NP}$-complete problem. Since $A \leqslant^p_T \bar{A}$ for any A, $\mathscr{NP} \cup co\mathscr{NP} \subseteq \Delta^p_2$. Moreover, $\mathscr{NP} \neq co\mathscr{NP}$ implies that $\mathscr{NP} \cup co\mathscr{NP}$ is properly contained in Δ^p_2.

5.6. PROPOSITION. *Assume $A \in \mathscr{NP} - co\mathscr{NP}$. Then $A \oplus \bar{A} \notin \mathscr{NP} \cup co\mathscr{NP}$. Moreover, for any B such that $A \oplus \bar{A} \leqslant^p_m B$, $B \notin \mathscr{NP} \cup co\mathscr{NP}$ too.*

Proof. Since $A \in \mathscr{NP} - co\mathscr{NP}$ iff $\bar{A} \in co\mathscr{NP} - \mathscr{NP}$, this follows from downward closure of $\mathscr{NP}$ and $co\mathscr{NP}$ under $\leqslant^p_m$.

In the following we assume $\mathscr{NP} \neq co\mathscr{NP}$, and we let $\mathscr{D} = \Delta^p_2 - (\mathscr{NP} \cup co\mathscr{NP})$ and $\mathbf{D} = \{\deg^p_T D: D \in \mathscr{D}\}$. Note that $\mathbf{D} \subseteq \mathbf{R}^p_T(\leqslant \mathbf{0}')$, $\mathbf{0}'$ the p–T-degree of $\mathscr{NP}$-complete problems, and, by Proposition 5.6, $A \oplus \bar{A} \in \mathscr{D}$ for any (w.r.t. $\leqslant^p_m$) $\mathscr{NP}$-complete set A. So $\mathbf{0}' \in \mathbf{D}$. It also follows from Proposition 5.6 that $\mathbf{D}$ is closed upwards in $\mathbf{R}^p_T(\leqslant \mathbf{0}')$. It is natural to ask whether there are degrees different from $\mathbf{0}'$ in $\mathbf{D}$, i.e., whether there are sets strictly p–T-reducible to an $\mathscr{NP}$-complete set which are neither in $\mathcal{NP}$ nor in $co\mathcal{NP}$. This question has been affirmatively answered by Schöning (1983) and Even, Long, and Yacobi (1982). This and further results on the structure of $\mathbf{D}$ can be obtained from our results on normal classes. Note that $\overline{\mathcal{NP} \cup co\mathcal{NP}}$ is normal and $\mathcal{D}$ is the restriction of $\overline{\mathcal{NP} \cup co\mathcal{NP}}$ to $\{B: B \leqslant^p_T A\}$, A some $\mathcal{NP}$-complete problem. So, since $\mathcal{D}$ is nonempty as observed above, the local version of Corollary 5.3 and Corollary 4.5 together with the fact that $\mathbf{D}$ is closed upwards in $\mathbf{R}^p_T(\leqslant \mathbf{0}')$ imply, e.g., the following: Any countable distributive lattice can be embedded in $\mathbf{D}$ preserving 1; for any p–T-degree $\mathbf{a}$ such that $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ there is a degree in $\mathbf{D}$ incomparable with $\mathbf{a}$; any degree in $\mathbf{D}$ is join reducible in $\mathbf{D}$, whence $\mathbf{D}$ has no minimal elements; any degree in $\mathbf{D} - \{\mathbf{0}'\}$ is meet reducible in $\mathbf{D}$, whence $\mathbf{D} - \{\mathbf{0}'\}$ has no maximal elements; $\mathbf{D}$ contains minimal pairs, whence $\mathbf{D}$ is not closed under $\cap$ and thus $\mathbf{D}$ is not a filter of $\mathbf{R}^p_T(\leqslant \mathbf{0}')$.

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